1.

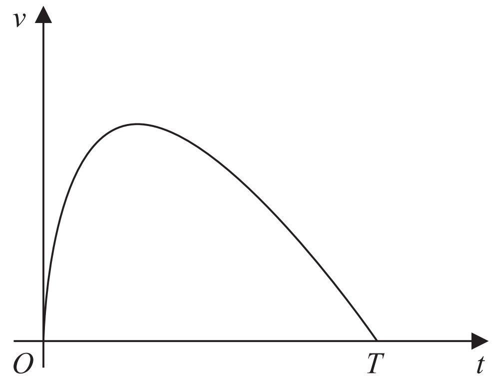


Figure 2

A car stops at two sets of traffic lights.

Figure 2 shows a graph of the speed of the car, $v\,\text{ms}^{-1}$, as it travels between the two sets of traffic lights.

The car takes T seconds to travel between the two sets of traffic lights.

The speed of the car is modelled by the equation

$$v = (10 - 0.4t)\ln(t + 1) \qquad 0 \leqslant t \leqslant T$$

where t seconds is the time after the car leaves the first set of traffic lights.

According to the model,

(a) find the value of T

(1)

(b) show that the maximum speed of the car occurs when

$$t = \frac{26}{1 + \ln(t + 1)} - 1$$

(4)

Using the iteration formula

$$t_{n+1} = \frac{26}{1 + \ln(t_n + 1)} - 1$$

with $t_1 = 7$

(c) (i) find the value of t_3 to 3 decimal places,

(ii) find, by repeated iteration, the time taken for the car to reach maximum speed.

(3)

(a) $(10 - 0.4t)\ln(t+1) = 0$ ← $v = 0$ when $t = 0$ and when $t = T$.

$10\ln(t+1) - 0.4t\ln(t+1) = 0$ $+ 0.4t\ln(t+1)$

$10\ln(t+1) = 0.4t\ln(t+1)$ $\div \ln(t+1)$ this is okay because we know $v = 0$ when $t = 0$, so $T > 0$. Then $T + 1 > 0$, so $\ln(t+1) \neq 0$.

$10 = 0.4t$

$25 = t$

$\therefore T = 25$ 1

(b) $v = (10 - 0.4t)\ln(t+1)$

let $v = f(t)g(t)$

then $v' = f(t)g'(t) + f'(t)g(t)$

$f(t) = 10 - 0.4t$ $f'(t) = -0.4$

$g(t) = \ln(t+1)$ $g'(t) = \frac{1}{t+1}$

$\frac{dv}{dt} = \ln(t+1) \times -0.4 + (10 - 0.4t) \times \frac{1}{t+1}$ 2

$0 = -0.4\ln(t+1) + \frac{10 - 0.4t}{t+1}$ 1 ← max speed when gradient is 0 (at turning point)

$\frac{10 - 0.4t}{t+1} = 0.4\ln(t+1)$ $\times (t+1)$

$10 - 0.4t = 0.4\ln(t+1) \times (t+1)$ $+ 0.4t$

$10 = 0.4t\ln(t+1) + 0.4\ln(t+1) + 0.4t$ $\div 0.4$

$25 = t\ln(t+1) + \ln(t+1) + t$ factorise

$25 = t(\ln(t+1) + 1) + \ln(t+1)$ $- \ln(t+1)$

$25 - \ln(t+1) = t(\ln(t+1) + 1)$ $\div (1 + \ln(t+1))$

$\frac{25 - \ln(t+1)}{1 + \ln(t+1)} = t$

$$\frac{26}{1+\ln(t+1)} - 1 = t \quad ①$$

(c) $$t_{n+1} = \frac{26}{1+\ln(t_n+1)} - 1$$

$t_1 = 7$

$$t_2 = \frac{26}{1+\ln(7+1)} - 1 \; ① = 7.298 \; ①$$

$$t_3 = \frac{26}{1+\ln(7.298+1)+1} = 7\,33 \quad ①$$

$t_3 = 7.33$ seconds

2. The function f is defined by

$$f(x) = \frac{e^{3x}}{4x^2 + k}$$

where k is a positive constant.

(a) Show that

$$f'(x) = (12x^2 - 8x + 3k)\,g(x)$$

where $g(x)$ is a function to be found.

(3)

Given that the curve with equation $y = f(x)$ has at least one stationary point,

(b) find the range of possible values of k.

(3)

a) $u = e^{3x}$ $\quad v = 4x^2 + k$

$u' = 3e^{3x}$ $\quad v' = 8x$ (1)

$f'(x) = \dfrac{v \times u' - u \times v'}{v^2}$

$= \dfrac{(4x^2+k)(3e^{3x}) - (e^{3x})(8x)}{(4x^2+k)^2}$ (1)

$= \dfrac{e^{3x}(12x^2 - 8x + 3k)}{(4x^2+k)^2}$

$= (12x^2 - 8x + 3k)\dfrac{e^{3x}}{(4x^2+k)^2}$ (1)

$\therefore g(x) = \dfrac{e^{3x}}{(4x^2+k)^2}$

$\therefore f'(x) = (12x^2 - 8x + 3k)\,g(x)$

b) $f'(x) = 0$: $(12x^2 - 8x + 3k)\,g(x) = 0$

As $g(x) \neq 0 \Rightarrow 12x^2 - 8x + 3k = 0$

Since the curve has at least one stationary point, then $12x^2 - 8x + k$ has at least one root. (1)

$b^2 - 4ac \geq 0$: $(-8)^2 - 4 \times 12 \times 3k \geq 0$ (1)

$64 - 144k \geq 0$

$k \leq \frac{4}{9}$

$\therefore 0 < k \leq \frac{4}{9}$ (1)

3. A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{7xe^x}{\sqrt{e^{3x} - 2}} \qquad x > \ln\sqrt[3]{2}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{7e^x(e^{3x}(2 - x) + Ax + B)}{2(e^{3x} - 2)^{\frac{3}{2}}}$$

where A and B are constants to be found.

(5)

(b) Hence show that the x coordinates of the turning points of the curve are solutions of the equation

$$x = \frac{2e^{3x} - 4}{e^{3x} + 4}$$

(2)

The equation $x = \dfrac{2e^{3x} - 4}{e^{3x} + 4}$ has two positive roots α and β where $\beta > \alpha$

A student uses the iteration formula

$$x_{n+1} = \frac{2e^{3x_n} - 4}{e^{3x_n} + 4}$$

in an attempt to find approximations for α and β

Diagram 1 shows a plot of part of the curve with equation $y = \dfrac{2e^{3x} - 4}{e^{3x} + 4}$ and part of the line with equation $y = x$

Using Diagram 1 on page 42

(c) draw a staircase diagram to show that the iteration formula starting with $x_1 = 1$ can be used to find an approximation for β

(1)

Use the iteration formula with $x_1 = 1$, to find, to 3 decimal places,

(d) (i) the value of x_2

(ii) the value of β

(3)

Using a suitable interval and a suitable function that should be stated

(e) show that $\alpha = 0.432$ to 3 decimal places.

(2)

a) $f(x) = \dfrac{7xe^x}{(e^{3x}-2)^{1/2}}$

$\dfrac{d}{dx}(7xe^x)$: let $u = 7x$ $\quad v = e^x$

$\dfrac{du}{dx} = 7 \quad \dfrac{dv}{dx} = e^x$

1 1

$\dfrac{du}{dx}v + \dfrac{dv}{dx}u = 7e^x + 7xe^x$

$\dfrac{d}{dx}\left((e^{3x}-2)^{1/2}\right) = \dfrac{1}{2} \times 3e^{3x} \times (e^{3x}-2)^{-1/2}$ 1

$= \dfrac{3}{2}e^{3x}(e^{3x}-2)^{-1/2}$

$f(x) = \dfrac{7xe^x}{(e^{3x}-2)^{1/2}}$

using quotient rule

1

$$f'(x) = \frac{(e^{3x}-2)^{1/2}(7e^x + 7xe^x) - 7xe^x\left(\frac{3}{2}e^{3x}(e^{3x}-2)^{-1/2}\right)}{e^{3x}-2}$$

$$= \frac{7(e^{3x}-2)^{-1/2}\left[e^x(e^{3x}-2)(1+x) - \frac{3}{2}xe^xe^{3x}\right]}{e^{3x}-2}$$

factoring out $7(e^{3x}-2)^{-1/2}$

$$= \frac{7e^x\left[(e^{3x}-2)(1+x) - \frac{3}{2}xe^{3x}\right]}{(e^{3x}-2)^{3/2}}$$

factoring out e^x

moving $(e^{3x}-2)^{-1/2}$ to the denominator

Only use the copy of Diagram 1 if you need to redraw your answer to part (c).

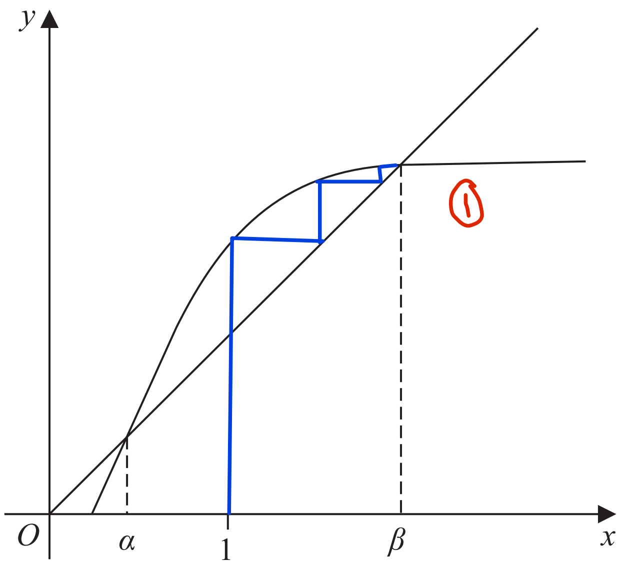
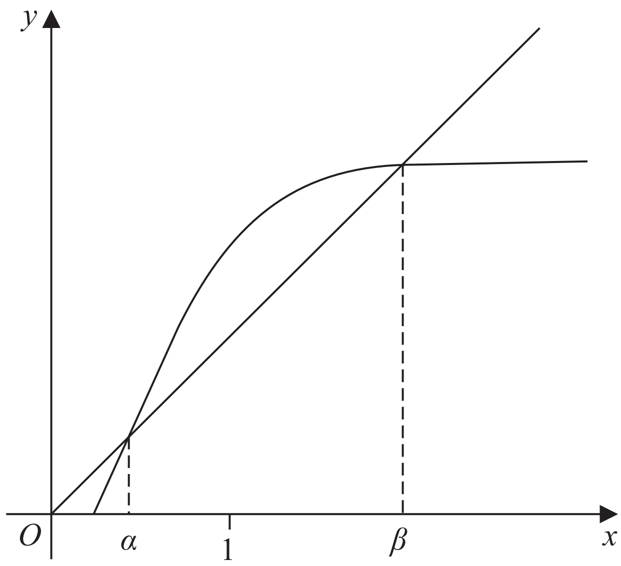


Diagram 1



copy of Diagram 1

expanding bracket

$$f'(x) = \frac{7e^x\left[e^{3x} + xe^{3x} - 2 - 2x - \frac{3}{2}xe^{3x}\right]}{(e^{3x}-2)^{3/2}}$$

collecting like terms

$$= \frac{7e^x\left[e^{3x} - \frac{1}{2}xe^{3x} - 2x - 2\right]}{(e^{3x}-2)^{3/2}}$$

multiplying top and bottom by 2

$$= \frac{7e^x(2e^{3x} - xe^{3x} - 4x - 4)}{2(e^{3x}-2)^{3/2}}$$

①

$$= \frac{7e^x(e^{3x}(2-x) - 4x - 4)}{2(e^{3x}-2)^{3/2}}$$ as required.

b) turning points have $f'(x)=0$

$$\Rightarrow \frac{7e^{x}\left(e^{3x}(2-x)-4x-4\right)}{2(e^{3x}-2)^{3/2}}=0$$

$e^{x}\neq 0$, multiply by $2(e^{3x}-2)^{3/2}$

$$e^{3x}(2-x)-4x-4=0$$

$$2e^{3x}-xe^{3x}-4x-4=0$$

$$x(e^{3x}+4)=2e^{3x}-4 \quad (1)$$

$$x=\frac{2e^{3x}-4}{e^{3x}+4} \quad (1)$$

c) drawn on diagram

d) $$x_{n+1}=\frac{2e^{3x_n}-4}{e^{3x_n}+4}$$

(i) $$x_2=\frac{2e^{3}-4}{e^{3}+4}=1.50177\ldots \quad (1)$$

$$=1.502 \text{ (3dp)} \quad (1)$$

(ii) $\beta = 1.96757\ldots$

$= 1.968$ (3dp) (1)

e) α is a solution of $x=\dfrac{2e^{3x}-4}{e^{3x}+4}$

$\therefore$ a solution of $\dfrac{2e^{3x}-4}{e^{3x}+4}-x=0$

so define $h(x) = \dfrac{2e^{3x}-4}{e^{3x}+4} - x$, so $h(\alpha)=0$.

$h(0.4315) = -0.000297\ldots < 0$
$h(0.4325) = 0.000947 > 0$ (1)

- since there is a change of sign
- and $h(x)$ is continuous
- $\alpha = 0.432$ (to 3dp) (1)